Solutions of fractional equations involving sources and Radon measures

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1 Introduction

In this note, we consider the existence of positive solution to

$$\begin{cases} (-\Delta)^\alpha u = u_+^p + \sigma\lambda, & \text{in} \quad \Omega, \\ u = 0, & \text{in} \quad \mathbb{R}^{\mathrm{N}} \setminus \Omega, \end{cases} \tag{1.1}$$

where $p > 0$, $\sigma > 0$, $\lambda \in \mathfrak{M}(\Omega)$ with $\mathfrak{M}(\Omega)$ the Radon measure space, $u_+(x) = \max\{u(x), 0\}$ and Ω is an open, smooth domain of $\mathbb{R}^N$ $(N \geq 2)$. Here $(-\Delta)^\alpha$ is defined, for a regular function u, as follow

$$(-\Delta)^\alpha u(x) = (\alpha - 1) \lim_{r \to 0} \int_{\mathbb{R}^N \setminus B_r} \frac{u(x+y) - u(x)}{|y|^{N+2\alpha}} dy, \tag{1.2}$$

where $\alpha \in (0,1)$, B_r denotes the ball centered at origin with radius r in $\mathbb{R}^N$. This definition is called *in the principle value sense.*

The original problem of (1.3) is

$$\begin{cases} -\Delta u = u_+^p + \sigma\lambda, & \text{in} \quad \Omega, \\ u = 0, & \text{in} \quad \partial\Omega, \end{cases} \tag{1.3}$$

which has been studied intensively, see [1, 2, 4, 19] for the existence.

In the study of elliptic equations involving Measures, the Green's functions plays an important role. Motivated by the construction of the Green's

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function for laplacian case, we first also consider the fundamental solution for fractional laplacian. We denote

$$\Gamma(x)=\frac{C(N,\alpha)}{|x|^{N-2\alpha}},\quad x\in\mathbb{R}^N\setminus\{0\},\tag{1.4}$$

where $C(N,\alpha)>0$ is such that

$$(-\Delta)^\alpha\Gamma=\delta_0$$

in the distribution sense. Also Γ is a fundamental solution to

$$(-\Delta)^\alpha u(x)=0,\quad x\in\mathbb{R}^N\setminus\{0\}.$$

in the principle value sense. The fundamental solution is the essential part to construct Green's function for fractional laplacian operator, that is,

Definition 1.1 *Let Ω be an open and smooth domain of $\mathbb{R}^N(N\ge 2)$. We denote*

$$G(x,y)=\Gamma(x-y)-\phi(x,y),\quad (x,y)\in\mathbb{R}^N\times\mathbb{R}^N\setminus D,\tag{1.5}$$

where Γ is defined by (1.4), $D=\{(z,z)\in\Omega\times\Omega\}$ and $\phi(x,y)$ is the solution of

$$\begin{cases}(-\Delta)_z^\alpha\phi(x,z)=0, & z\in\Omega,\\ \phi(x,z)=\Gamma(x-z), & z\in\mathbb{R}^N\setminus\Omega,\end{cases}\tag{1.6}$$

for any given $x\in\Omega$, and

$$\begin{cases}(-\Delta)_z^\alpha\phi(z,y)=0, & z\in\Omega,\\ \phi(z,y)=\Gamma(z-y), & z\in\mathbb{R}^N\setminus\Omega.\end{cases}\tag{1.7}$$

for any given $y\in\Omega$.

We call $G(x,y)$ as Green's function to fractional laplacian with order α.

It is known that the study of the elliptic equations involving measures is based on the estimate of singular behavior of Green's Function, see [7]. Equation (1.3) involving Radon measure, of course, it isn't supposed to find a regular solution. So one type of weak solution of (1.3) by Green's function for fractional laplacian operator should be introduced, there is,

Definition 1.2 *Let Ω be an open and smooth domain of $\mathbb{R}^N(N\ge 2)$ and $G(x,y)$ is the Green function. We say u is a weak solution of (1.3) if u is measurable, $\mathbb{G}(u_+^p)\in L^1(\Omega)$ and*

$$u(x)=\mathbb{G}(u_+^p)(x)+\sigma\mathbb{G}(\lambda)(x),\quad \text{a.e. } \mathrm{x}\in\mathrm{\mathbb{R}^N},\tag{1.8}$$

where $\mathbb{G}(u_+^p)(x)=\int_\Omega G(x,y)u_+^p(y)dy$ and $\mathbb{G}(\lambda)(x)=\int_\Omega G(x,y)d\lambda(y)$ with G being the Green's function for fractional laplacian by Definition 1.1.

We remark here that by the definition of G, we know that for $x \in \mathbb{R}^N \setminus \Omega$, $G(x,y) = 0$ for any $y \in \mathbb{R}^N$, so

$$u(x) = 0 \quad x \in \mathbb{R}^N \setminus \Omega.$$

Now we are in the position to show the main existence theorem for fractional equation involving measures and reaction source.

Theorem 1.1 *Assume that Ω is an open, bounded and smooth domain of $\mathbb{R}^N$ and $\lambda \in \mathfrak{M}(\Omega)$. Then there exists a solution $u \in L^1_{loc}(\Omega)$ of (1.3) in the sense of (1.8) for σ small enough, if one of the following assumptions:*

(i) $1 < p < \frac{N}{N-2\alpha}$;

(ii) $p > \frac{N}{N-2\alpha}$ and $\lambda_+ \in \mathrm{L^q}(\Omega)$ with $\mathrm{q} \geq \frac{\mathrm{Np}}{\mathrm{N+2\alpha p}}$;

(iii) $p = \frac{N}{N-2\alpha}$ and $\lambda_+ \in \mathrm{L^q}(\Omega)$ with $\mathrm{q} > 1$.

Moreover, if $\int_\Omega G(x,y)d\lambda(y) \geq 0$ a.e. $x \in \Omega$, then $u \geq 0$.

Remark 1.1 *We see that*

$$\frac{Np}{2\alpha p + N} = 1 \quad \text{if} \quad \mathrm{p} = \frac{\mathrm{N}}{\mathrm{N} - 2\alpha}.$$

In particular, for $\lambda = \delta_{x_0}$ the Dirac mass at point $x_0 \in \Omega$, Theorem 1.1 gives the existence of solution to (1.3) for $p \in (1, \frac{N}{N-2\alpha})$. In what follows, our interest is to study the asymptotic behavior of the solution near x_0 with $p \in (0, \frac{N}{N-2\alpha})$. The asymptotic behavior the solution for $p \in (1, \frac{N}{N-2\alpha})$ is stated as:

Theorem 1.2 *Suppose that Ω is an open, bounded and smooth domain of $\mathbb{R}^N$ ($N \geq 2$) and $1 < p < \frac{N}{N-2\alpha}$. There exists $\sigma_0 > 0$ such that for any $\sigma \in (0, \sigma_0]$, problem (1.3) with $\lambda = \delta_{x_0}$ admits a solution u, satisfying that for $x \in B_\epsilon(x_0)$ with $\epsilon \in (0, \frac{\min\{d(x_0),1\}}{4})$,*
(i) if $p > \frac{2\alpha}{N-2\alpha}$, then

$$\frac{\sigma^p C^{-1}}{|x-x_0|^{(N-2\alpha)p-2\alpha}} < u(x) - \frac{\sigma C(N,\alpha)}{|x-x_0|^{N-2\alpha}} \leq \frac{\sigma^p C}{|x-x_0|^{(N-2\alpha)p-2\alpha}},$$

(ii) if $p = \frac{2\alpha}{N-2\alpha}$, then

$$-\sigma^p C^{-1} \ln(|x-x_0|) < u(x) - \frac{\sigma C(N,\alpha)}{|x-x_0|^{N-2\alpha}} \leq -\sigma^p C \ln(|x-x_0|),$$

(iii) if $p < \frac{2\alpha}{N-2\alpha}$, *then*

$$\sigma^p C^{-1} < u(x) - \frac{\sigma C(N,\alpha)}{|x-x_0|^{N-2\alpha}} \le \sigma^p C,$$

where $C(N,\alpha)$ *is from (1.4) and* $C > 1$ *depends on* N, α, Ω *and* x_0.

For the case $p \in (0,1)$, we have

Theorem 1.3 *Suppose that* Ω *is an open, bounded and smooth domain of* $\mathbb{R}^N$ $(N \ge 2)$ *and* $0 < p < 1$. *Then for any* $\sigma > 0$, *problem (1.3) with* $\lambda = \delta_{x_0}$ *admits a solution* u, *satisfying that for* $x \in B_\epsilon(x_0)$ *with* $\epsilon \in (0, \frac{\min\{d(x_0),1\}}{4})$,
(i) if $p > \frac{2\alpha}{N-2\alpha}$, *then*

$$\frac{\sigma^p C^{-1}}{|x-x_0|^{(N-2\alpha)p-2\alpha}} < u(x) - \frac{\sigma C(N,\alpha)}{|x-x_0|^{N-2\alpha}} \le \frac{(C+\sigma^{1-p})^{\frac{p}{1-p}}}{|x-x_0|^{(N-2\alpha)p-2\alpha}},$$

(ii) if $p = \frac{2\alpha}{N-2\alpha}$, *then*

$$-\sigma^p C^{-1}\ln(|x-x_0|) < u(x) - \frac{\sigma C(N,\alpha)}{|x-x_0|^{N-2\alpha}} \le -(C+\sigma^{1-p})^{\frac{p}{1-p}}\ln(|x-x_0|),$$

(iii) if $p < \frac{2\alpha}{N-2\alpha}$, *then*

$$\sigma^p C^{-1} < u(x) - \frac{\sigma C(N,\alpha)}{|x-x_0|^{N-2\alpha}} \le (C+\sigma^{1-p})^{\frac{p}{1-p}},$$

where $C(N,\alpha)$ *is from (1.4) and* $C > 1$ *depends on* N, α, Ω *and* x_0.

For $p = 1$, problem (1.3) may non-exist solution for any $\sigma > 0$. See an example with $\alpha = 1$, let λ_1 and ϕ_1 be the first eigenvalue and eigenfunction respectively of

$$\begin{cases} -\Delta u = \lambda_1 u_+, & \text{in} \quad \Omega, \\ u(x) = 0, & \text{on} \quad \partial\Omega. \end{cases}$$

In particular, for some type domain Ω, it could be $\lambda_1 \le 1$. If

$$\begin{cases} -\Delta u = u_+ + \sigma\delta_{x_0}, & \text{in} \quad \Omega, \\ u(x) = 0, & \text{on} \quad \partial\Omega. \end{cases}$$

admits a positive solution, then by computing directly, we have that

$$\phi_1(x_0) = 0,$$

which is impossible with $\phi_1 > 0$ in Ω.

So in the case of $p = 1$, we consider the following problem

$$\begin{cases} (-\Delta)^\alpha u = \Lambda u_+ + \sigma\lambda, & \text{in} \quad \Omega, \\ u(x) = 0, & \text{in} \quad \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.9}$$

where $\Lambda > 0$.

Similarly to Definition 1.2, we say that u is a weak solution of (1.9) if $u \in L^1(\Omega)$ and

$$u(x) = \Lambda\mathbb{G}(u_+^p)(x) + \sigma\mathbb{G}(\lambda)(x), \quad \text{a.e. } x \in \mathbb{R}^N. \tag{1.10}$$

Theorem 1.4 *Assume that Ω is an open, smooth and bounded domain of $\mathbb{R}^N$. Then there exists $\Lambda_0 > 0$ such that for any $\Lambda \in (0, \Lambda_0)$, for any $\sigma > 0$, problem (1.9) with $\lambda = \delta_{x_0}$ admits a solution u, satisfying that for $x \in B_\epsilon(x_0)$ with $\epsilon \in (0, \frac{\min\{d(x_0),1\}}{4})$,*
(i) if $\frac{2\alpha}{N-2\alpha} < 1$, then

$$\frac{\sigma\Lambda C^{-1}}{|x - x_0|^{N-4\alpha}} < u(x) - \frac{\sigma C(N, \alpha)}{|x - x_0|^{N-2\alpha}} \le \frac{\sigma\Lambda C}{|x - x_0|^{N-4\alpha}},$$

(ii) if $\frac{2\alpha}{N-2\alpha} = 1$, then

$$-\sigma\Lambda C^{-1} \ln(|x - x_0|) < u(x) - \frac{\sigma C(N, \alpha)}{|x - x_0|^{N-2\alpha}} \le -\sigma\Lambda C \ln(|x - x_0|),$$

(iii) if $\frac{2\alpha}{N-2\alpha} > 1$, then

$$\sigma\Lambda C^{-1} < u(x) - \frac{\sigma C(N, \alpha)}{|x - x_0|^{N-2\alpha}} \le \sigma\Lambda C,$$

where $C(N, \alpha)$ is from (1.4) and $C > 1$ depends on N, α, Ω and x_0.

Moreover, the solution is unique.

In the Theorem 1.2 and Theorem 1.4, the singular estimate is more precisely stated than

$$u(x) = \frac{\sigma C(N, \alpha)}{|x - x_0|^{N-2\alpha}}(1 + o(1)).$$

This article is organized as follows. In section §2 we present some preliminaries to the Green's function. Section §3 is devoted to obtain the existence of solution to (1.3) with general convex reaction sources by Conjugate method. In section §4 we prove Theorem 1.1 by applying the results of Section §3. Finally, Theorem 1.2 is shown in section §5.

2 Green's function for Fractional Laplacian

In this section, we consider the properties of Green's function for fractional laplacian operator. Motivated by local operator Δ, the Green's function with order α could be used to solve the Dirichlet type problem involving fractional laplacian. And its representation formula using Green's function is stated as:

Theorem 2.1 *Assume that Ω is an open and smooth domain of $\mathbb{R}^N$, $f \in \mathcal{S}$, where $\mathcal{S}$ is the Schwartz space of rapidly decaying C^∞ functions in $\mathbb{R}^N$ and the Green function G is defined by (1.5).*
Then

$$u(x) = \int_\Omega G(x,y)f(y)dy \tag{2.1}$$

is the solution of

$$\begin{cases} (-\Delta)^\alpha u = f & \text{in} \quad \Omega, \\ u = 0 & \text{in} \quad \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.2}$$

In order to prove Theorem 2.1, let's study first about the fundamental solution Γ.

Lemma 2.1 *Let $f \in \mathcal{S}$. Then*

$$u(x) = \Gamma * f = \int_{\mathbb{R}^N} \Gamma(x-y)f(y)dy$$

is the solution of

$$(-\Delta)^\alpha u = f \quad \text{in} \quad \mathbb{R}^N. \tag{2.3}$$

Proof. In fact, From Proposition 3.3 in [10], for $(-\Delta)^\alpha$ defined (1.2) and $u \in \mathcal{S}$

$$(-\Delta)^\alpha u = \mathcal{F}^{-1}(|\xi|^{2\alpha}\mathcal{F}u) \quad \text{in} \quad \mathbb{R}^N.$$

Then we obtain that

$$\begin{aligned} u(x) &= \mathcal{F}^{-1}(\tfrac{1}{|\xi|^{2\alpha}}\mathcal{F}(f)) \\ &= \mathcal{F}^{-1}(\tfrac{1}{|\xi|^{2\alpha}}) * f \end{aligned} \tag{2.4}$$

We first claim that

$$\Gamma(x) = \mathcal{F}^{-1}(\frac{1}{|\xi|^{2\alpha}}). \tag{2.5}$$

Assume (2.5) holds at this moment, then (2.4) turns to be

$$u = \Gamma * f.$$

Now we prove (2.5). To this end, we start defining the heat kernel, for $\alpha \in (0,1)$ and $x \in \mathbb{R}^N$, as

$$\mathcal{H}(x,t) = \int_{\mathbb{R}^N} e^{2\pi x\cdot\xi - t|\xi|^{2\alpha}} d\xi \tag{2.6}$$

and denote

$$\mathcal{K}(x) = C\int_0^\infty \mathcal{H}(x,t)dt, \tag{2.7}$$

where $C > 0$ will choose later.

Step 1. To prove there exists $C > 0$ in (2.7) such that

$$\mathcal{K} = \mathcal{F}^{-1}(\frac{1}{|\xi|^{2\alpha}}).$$

Indeed, letting $\xi \in \mathbb{R}^N \setminus \{0\}$ and changing variables $x = \frac{\tilde{x}}{|\xi|}$ and $z = |\xi|\tilde{z}$ and $t = \frac{\tilde{t}}{|\xi|^{2\alpha}}$, for simplicity, still denoting $\tilde{z}$, $\tilde{t}$ and $\tilde{x}$ by z, t and x,

$$\begin{aligned}\mathcal{F}(\mathcal{K})(\xi) &= \int_{\mathbb{R}^N} e^{-2\pi i\xi\cdot x}\int_0^\infty\int_{\mathbb{R}^N} e^{2\pi x\cdot z - t|z|^{2\alpha}} dzdtdx \\ &= \frac{1}{|\xi|^{2\alpha}}\int_{\mathbb{R}^N} e^{-2\pi i\vec{e}_\xi\cdot x}\int_0^\infty\int_{\mathbb{R}^N} e^{2\pi x\cdot z - t|z|^{2\alpha}} dzdtdx\end{aligned}$$

where $\vec{e}_\xi = \frac{\xi}{|\xi|}$.

Denote $C(\vec{e}_\xi) = \int_{\mathbb{R}^N} e^{-2\pi i\vec{e}_\xi\cdot x}\int_0^\infty\int_{\mathbb{R}^N} e^{2\pi x\cdot z - t|z|^{2\alpha}} dzdtdx$. We claim first that for $|\vec{e}_x| = |\vec{e}_y| = 1$,

$$C(\vec{e}_x) = C(\vec{e}_y).$$

In fact, there exists a matrix A with $|A| = 1$ such that $\vec{e}_x = A\vec{e}_y$, and then by changing variable, we have the claim. Now we can let $\vec{e}_\xi = (1, 0, \cdots, 0)$ and then

$$C(\vec{e}_\xi) = \int_{\mathbb{R}^N} e^{-2\pi i x_1}\int_0^\infty\int_{\mathbb{R}^N} e^{2\pi x\cdot z - t|z|^{2\alpha}} dzdtdx > 0.$$

Step 2. To prove

$$\mathcal{K} = \Gamma.$$

For $x \in \mathbb{R}^N \setminus \{0\}$, by changing variables $\tilde{\xi} = |x|\xi$ and $\tilde{t} = \frac{t}{|x|^{2\alpha}}$

$$\begin{aligned}\mathcal{K}(x) &= \int_0^\infty\int_{\mathbb{R}^N} e^{2\pi x\cdot\xi - t|\xi|^{2\alpha}} d\xi dt \\ &= \frac{1}{|x|^{N-2\alpha}}\int_0^\infty\int_{\mathbb{R}^N} e^{2\pi\vec{e}_x\cdot\tilde{\xi} - t|\tilde{\xi}|^{2\alpha}} d\tilde{\xi}d\tilde{t},\end{aligned}$$

where $\vec{e}_x = \frac{x}{|x|}$. Denote

$$\tilde{C}_x = \int_0^\infty \int_{\mathbb{R}^N} e^{2\pi \vec{e}_x \cdot \tilde{\xi} - t|\tilde{\xi}|^{2\alpha}} d\tilde{\xi} d\tilde{t}.$$

Similarly to step 1, $\tilde{C}_x$ is some positive constant independent of x. By choose $C(N,\alpha) = \tilde{C}_x$ in (1.4), then $\mathcal{K} = \Gamma$. And we finish the proof. $\square$

Remark 2.1 *From Lemma 2.1, the fundamental solution* Γ *could be seen as*

$$(-\Delta)^\alpha \Gamma(\cdot) = \delta_0, \tag{2.8}$$

in the distribution sense.

Proof of Theorem 2.1. Since Ω is smooth and $f \in \mathcal{S}$. For $x \in \Omega$,

$$(-\Delta)^\alpha \Gamma(x-y) = 0, \quad y \in \mathbb{R}^N \setminus \Omega.$$

For u defined by (2.1) and $x \in \Omega$,

$$\begin{aligned}
(-\Delta)^\alpha u(x) &= (\alpha - 1) \lim_{r \to 0} \int_{\mathbb{R}^N \setminus B_r} \frac{\int_\Omega G(x+z,y) f(y) dy - \int_\Omega G(x,y) f(y) dy}{|z|^{N+2\alpha}} dz \\
&= \int_\Omega (-\Delta)_x^\alpha G(x,y) f(y) dy \\
&= \int_\Omega (-\Delta)^\alpha \Gamma(x-y) f(y) dy - \int_\Omega (-\Delta)_x^\alpha \phi(x,y) f(y) dy \\
&= \int_{\mathbb{R}^N} (-\Delta)^\alpha \Gamma(x-y) f(y) dy \\
&= f(x),
\end{aligned}$$

the last equality used Remark 2.1. $\square$

Remark 2.2 *From Theorem1.1, the Green's function is the solution of*

$$\begin{cases} (-\Delta)^\alpha G(x,\cdot) = \delta_x, & \text{in the distribution sense,} \\ G(x,y) = 0, & y \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2.9}$$

for any $x \in \Omega$.

Remark 2.3 *It is well-known that it is studied in many papers, such as [7, 6], for singular behavior of Green's Function expressed by transition density which is equivalent to Definition 1.1, by remark 2.2.*

Theorem 2.2 *Assume that Ω is an open and smooth domain of $\mathbb{R}^N$ and $d(x) = dist(x, \mathbb{R}^N \setminus \Omega)$ for $x \in \Omega$. Then there exists $C > 1$ dependent of N, α such that*

$$\begin{aligned} C^{-1} \min\{\tfrac{1}{|x-y|^{N-2\alpha}}, \tfrac{d^\alpha(x)d^\alpha(y)}{|x-y|^N}\} &\leq G(x,y) \\ &\leq C \min\{\tfrac{1}{|x-y|^{N-2\alpha}}, \tfrac{d^\alpha(x)d^\alpha(y)}{|x-y|^N}\} \end{aligned} \tag{2.10}$$

for $(x,y) \in \Omega \times \Omega \setminus D$ with D defined in Definition 1.1, and

$$\frac{G(x,y)G(y,z)}{G(x,z)} \leq C \frac{|x-z|^{N-2\alpha}}{|x-y|^{N-2\alpha}|y-z|^{N-2\alpha}}, \tag{2.11}$$

for $(x,y), (y,z), (x,z) \in \Omega \times \Omega \setminus D$.

Proof. The inequality (2.10) and (2.11) see Corollary 1.3 and Theorem 1.6 (3G Theorem), respectively, in [7]. $\square$

Theorem 2.3 *Assume that Ω is an open and smooth domain of $\mathbb{R}^N$. Then*

$$0 < \phi(x,y) < C(N,\alpha) \min\{d(x)^{2\alpha-N}, d(y)^{2\alpha-N}\} \tag{2.12}$$

for $(x,y) \in \Omega \times \Omega$, and

$$G(x,y) = 0, \quad \text{if} \quad \mathrm{x} \in \mathbb{R}^\mathrm{N} \setminus \Omega \text{ or } \mathrm{y} \in \mathbb{R}^\mathrm{N} \setminus \Omega.$$

Moreover,

$$G(x,y) = G(y,x), \quad x, y \in \Omega, \ x \neq y. \tag{2.13}$$

Proof. We divide the proof into several steps.
Step 1: Prove that fixed $x \in \Omega$, $\phi(x,y) < \frac{C(N,\alpha)}{d(x)^{N-2\alpha}}$ for any $y \in \Omega$. If not, there exits a point $y_0 \in \Omega$ such that

$$\phi(x,y_0) = \max_{y \in \Omega} \phi(x,y) \geq \frac{C(N,\alpha)}{d(x)^{N-2\alpha}}.$$

Therefore, we have

$$(-\Delta)_y^\alpha \phi(x,y_0) = (\alpha - 1) \int_{\mathbb{R}^N} \frac{\phi(x,y_0+z) - \phi(x,y_0)}{|z|^{N+2\alpha}} dz > 0$$

which contradicts $(-\Delta)_y^\alpha \phi(x,y_0) = 0$, obtained by the definition of ϕ.

Similarly, we have $\phi(x,y) < \frac{C(N,\alpha)}{d(y)^{N-2\alpha}}$.

Step 2: We prove that $\phi(x,y) > 0$ in $(\Omega \times \Omega)$. If not, there exists $(x_0, y_0) \in \Omega \times \Omega$ such that $\phi(x_0, y_0) \le 0$. Then there exists $\bar{y} \in \Omega$ such that $\phi(x_0, \bar{y}) = \min_{y\in\Omega} \phi(x_0, y) \le 0$. Therefore,

$$(-\Delta)^\alpha_y \phi(x_0, \bar{y}) = (\alpha - 1) \int_{\mathbb{R}^N} \frac{\phi(x_0, \bar{y} + z) - \phi(x_0, \bar{y})}{|z|^{N+2\alpha}} dz < 0,$$

which contradicts $(-\Delta)^\alpha_y \phi(x_0, y_0) = 0$, obtained by the definition of ϕ.
Step 3: We prove (2.13), in fact, by the fact of

$$\Gamma(x - z) = \Gamma(z - x), \quad z \neq x, \tag{2.14}$$

we just have to prove that

$$\phi(x, z) = \phi(z, x).$$

In fact, put (2.14) into (1.6) and (1.7) with $x = y$. Then we have that

$$\phi(x, z) = \phi(z, x),$$

for any $z \in \mathbb{R}^N$. And we finish the proof. $\square$

3 Equations with general reaction sources

In this section, we study the existence results to equations with general reaction sources, that is,

$$\begin{cases} (-\Delta)^\alpha u = j(x, u) + \lambda, & \text{in} \quad \Omega, \\ u(x) = 0, & \text{in} \quad \mathrm{R^N} \setminus \Omega, \end{cases} \tag{3.1}$$

where Ω is open and smooth domain in $\mathbb{R}^N$.

Let $K_n = \Omega_n \cap \bar{B}_{R_n}$, where B_{R_n} is the ball centered at the original with radius $R_n > 1$, R_n strictly to n and $R_n \to +\infty$ as $n \to +\infty$, and $\Omega_n = \{x \in \Omega, dist(x, \partial\Omega) \le \frac{1}{R_n}\}$. Then

$$\forall n \ge 0, \quad |K_n| < \infty, \quad \bigcup_{n\ge1} K_n = \Omega.$$

We call following assumption as Conjugate-Condition

(*i*) $r \to j(x, r)$ is nondecreasing, convex and lower semi-continuous for a.e. $x \in \Omega$;

(ii) $j(x,0)=0$, a.e. in Ω;

The conjugate function j^*, defined by

$$j^*(x,r)=\sup_{a\in\mathbb{R}}(ra-j(x,r)).$$

Then j^* satisfies Conjugate-Condition. For simplicity,

$$j(u)(x)=\begin{cases} j(x,u(x)), & \text{if} \quad \mathrm{u(x)}<\infty, \\ \lim_{r\to\infty} j(x,u), & \text{if} \quad \mathrm{u(x)}=\infty \end{cases} \tag{3.2}$$

and $j^*(u)$ is similarly defined.

We denote

$$\mathbb{G}(h)(x)=\int_\Omega G(x,y)h(y)dy,$$

where $G(x,y)$ is a Green's function in Ω. By (2.13),

$$\mathbb{G}^*(h)(y)=\int_\Omega G(x,y)h(x)dx=\mathbb{G}(h)(y).$$

In particular,

$$f(x)=\int_\Omega G(x,y)d\lambda(y). \tag{3.3}$$

We also denote

$$L^\infty_{c+}(\Omega)=L^\infty_c(\Omega)\cap L_+(\mathbb{R}^N),$$

where $L^\infty_c(\Omega)=\{\xi:\mathbb{R}^N\to\mathbb{R},\ supp(\xi)\subset K_n,\ n \text{ big enough and } \operatorname{ess\,sup}|\xi|<+\infty\}$ and $L_+(\mathbb{R}^N)$ is the space of nonnegative measurable functions.

Being given $C\geq 1$ and $h\in L^\infty_{c+}(\Omega)$, we denote

$$F_C(h)=\begin{cases} \int_\Omega j^*(\frac{Ch}{\mathbb{G}(h)})\mathbb{G}(h)d\mu, & \text{if} \quad \frac{\mathrm{h}}{\mathbb{G}(\mathrm{h})}<+\infty \text{ a.e.} \\ & \qquad \text{and} \quad \mathrm{j}^*(\frac{\mathrm{Ch}}{\mathbb{G}(\mathrm{h})})\mathbb{G}(\mathrm{h})\in \mathrm{L}^1(\Omega), \\ +\infty, & \text{if not.} \end{cases} \tag{3.4}$$

with the convention $F(h)=F_C(h)$ if $\mathrm{C}=1$, $\frac{\mathrm{h}}{\mathbb{G}(\mathrm{h})}=0$ if $\mathrm{h}=\mathbb{G}(\mathrm{h})=0$, and $uh=0$ if $\mathrm{h}=0$ and $u=\infty$.

We put

$$\mathbb{X}=\{h\in L^\infty_{c+}(\Omega): \quad F(h)<\infty\} \tag{3.5}$$

and

$$\hat{\mathbb{X}}=\{h\in L^\infty_{c+}(\Omega): \quad \exists C>1 \ s.t.\ F_C(h)<\infty\}.$$

Theorem 3.1 *Assume that $f \geq 0$ is measurable. Then (3.1) admits a solution in the sense of*

$$\begin{cases} (i)\ u \in L_+(\Omega), \quad u(x) = \mathbb{N}(j(u))(x) + f(x), \quad \text{a.e} \ \text{ in } \mathbb{R}^{\mathrm{N}}, \\ (ii)\ uh \in L^1(\mathbb{R}^N), \quad \forall h \in \hat{\mathbb{X}}, \end{cases} \tag{3.6}$$

if and only if

$$\int_\Omega fh d\mu \leq F(h), \quad \forall h \in \hat{\mathbb{X}}. \tag{3.7}$$

Proof. We first prove $'' \Longrightarrow''$.
Multiplying (4.2) by $h \in L^\infty_{c+}$ and integrating over Ω implies that

$$\begin{aligned} \int_\Omega fh dx &= \int_\Omega (u - \mathbb{G}(j(u)))h dx \\ &= \int_\Omega (uh - j(u)\mathbb{G}(h))dx \\ &= \int_\Omega [u\frac{h}{\mathbb{G}(h)} - j(u)]\mathbb{G}(h))dx \\ &\leq \int_\Omega j^*(\frac{h}{\mathbb{G}(h)})\mathbb{G}(h)dx \\ &= F(h). \end{aligned}$$

Now we prove that $'' \Longleftarrow''$.
We denote that

$$\mathbb{G}_m(h) = \int_\Omega \min\{\chi_{K_m}(x)\chi_{K_m}(y)G(x,y), m\}h(y)dy, \quad m \in \mathbb{N}$$

and

$$f_n(x) = \min\{\chi_{K_n}(x)f(x), n\}, \quad m \in \mathbb{N}.$$

Fix m, we define that

$$\begin{cases} (i)\ u_0 = \mu f_0, \\ (ii) \forall n \geq 0 \quad \beta_n = \inf\{\chi_{K_n} j'(u_n), n\} \\ (iii) u_{n+1} = \mu\mathbb{G}_m(u_n\beta_n - j^*(\beta_n)) + \mu f_n, \end{cases} \tag{3.8}$$

where $\mu \in [0,1]$, $m, n \in \mathbb{N}$. In fact, u_n and β_n depend on n and μ, so we note $u_n = u(n,m,\mu)$ and $\beta_n = \beta(n,m,\mu)$.

Then for any $n \geq 1$ $m \geq 0$ and $\mu \in (0,1)$,

$$u_n \leq u_{n+1}, \quad \beta_n \leq \beta_{n+1} \tag{3.9}$$

and $m \to u(n,m,\mu)$, $\mu \to u(n,m,\mu)$ are increasing strictly for n fixed.
Step 1. Now we prove that for any $n,\ m$ and $\mu \in (0,1)$,

$$\int_\Omega u_n h dx \le \frac{\mu}{1-\mu} F(h), \quad h \in \hat{\mathbb{X}}. \tag{3.10}$$

Claim 1. For any $h \in \hat{\mathbb{X}}$, there exists $C \in (1, \frac{1}{\mu})$ such that $F_C(h) < \infty$. Then there exists $\phi_n \in L_c^\infty(\Omega)$ is the solution of

$$\phi_n = \max\{\frac{1}{C}\beta_n G_m(\phi_n), h\}. \tag{3.11}$$

We assume Claim 1 is right at this moment, and we continue to prove (3.10). We observe that $\phi_n \in L_c^\infty(\Omega)$ and

$$F_C(\phi_n) \le \int_\Omega j^*(\frac{\max\{\beta_n \mathbb{G}_m(\phi_n), Ch\}}{\mathbb{G}(\phi_n)})\mathbb{G}(\phi_n)dx.$$

Since j^* is increasing, then

$$F_C(\phi_n) \le \int_\Omega \max\{j^*(\frac{\beta_n \mathbb{G}_m(\phi_n)}{\mathbb{G}(\phi_n)}), j^*(\frac{Ch}{\mathbb{G}(\phi_n)})\}\mathbb{G}(\phi_n)dx.$$

By using the fact of $\mathbb{G} \ge \mathbb{G}_m$, $\phi_n \ge h$ and $j^*(ar) \le aj^*(r)$, $a \in [0,1]$,

$$\begin{aligned} F_C(\phi_n) &\le \textstyle\int_\Omega \max\{j^*(\beta_n)\mathbb{G}_m(\phi_n), j^*(\frac{Ch}{\mathbb{G}(h)})\mathbb{G}(h)\}dx \\ &\le \textstyle\int_\Omega j^*(\beta_n)\mathbb{G}_m(\phi_n)dx + F_C(h). \end{aligned} \tag{3.12}$$

Since $\mathbb{G}_m(\phi_n) \le m|B_{R_m}|\|\phi_n\|_{L^\infty}$, then $j^*(\beta_n) \le u_n\beta_n \in L_0^\infty(\Omega)$.

Multiplying (3.8)part (iii) by ϕ_n and integrating over Ω implies that

$$\begin{aligned} \int_\Omega u_{n+1}\phi_n &= \mu \int_\Omega [u_n\beta_n - j^*(\beta_n)]\mathbb{G}_m(\phi_n)dx + \mu \int_\Omega f_n\phi_n dx \\ \text{by (3.7)} &\le \mu \int_\Omega [u_n\beta_n - j^*(\beta_n)]\mathbb{G}_m(\phi_n)dx + \mu F_C(\phi_n) \\ \text{by (3.12)} &\le \mu \int_\Omega u_n\beta_n \mathbb{G}_m(\phi_n)dx + \mu F_C(h) \\ &\le \mu C \int_\Omega u_n\phi_n dx + \mu F_C(h) \\ &\le \mu C \int_\Omega u_{n+1}\phi_n dx + \mu F_C(h), \end{aligned}$$

that is,

$$\int_\Omega u_{n+1}\phi_n dx \le \frac{\mu}{1-\mu C},$$

and make $C \to 1$ to get our results.
Step 2 convergence. By Monotone and step 1, we have that

$$u_n \to u(m,\mu) \text{ as n} \to +\infty,$$

$$f_n \to f \text{ as n} \to +\infty$$

and

$$\begin{aligned} u_n\beta_n - j^*(\beta_n) &\to u(m,\lambda)j'(u(m,\lambda)) - j^*(j'(u(m,\lambda))) \\ &= j(u(m,\lambda)) \text{ as n} \to +\infty. \end{aligned}$$

We see that $u(m,\mu) = \mu\mathbb{G}_m(j(u(m,\mu)))(x) + f(x)$ a.e in Ω, then make $m \to +\infty$, $u(m,\mu) \to u_\mu$ such that $u_\mu \geq 0$ measurable, $u_\mu h \in L^1(\Omega)$ for any $h \in \hat{\mathbb{X}}$ and

$$u_\mu = \mu\mathbb{G}(j(u_\mu))(x) + f(x) \quad \text{a.e. in } \Omega.$$

This implies, in particular,

$$\int_\Omega u_\mu h dx = \mu\int_\Omega j(u_\mu)\mathbb{G}(h)dx + \mu\int_\Omega fhdx, \quad h \in \hat{\mathbb{X}}.$$

For $C > 1$ such that $F_C(h) < \infty$, then

$$\mu\int_\Omega u_\mu(\frac{Ch}{\mathbb{G}(h)} - j(u_\mu)\mathbb{G}(h)dx = (\mu C - 1)\int_\Omega u_\mu hdx + \mu\int_\Omega fhdx,$$

and consequently

$$\int_\Omega u_\mu hdx \leq \frac{\mu}{C\mu - 1}F_C(h). \tag{3.13}$$

Put $\mu \to 1$, by monotone of $\mu \to u_\mu$ and (3.13), then there exists $u \geq 0$ measurable such that $uh \in L^1(\Omega)$ and u is the solution of (4.2).
Proof of Claim 1. For n and m fixed, let

$$A_n\varphi = \beta_n\mathbb{G}_m\varphi,$$

with r_n spectral radius and we assume, at this moment, that

$$r_n < C \quad \text{where C independent of n.} \tag{3.14}$$

We see that

$$(CI - A_n)^{-1} = \sum_{i\geq 0} C^{-(i+1)}A_n^i \quad \text{in } \mathrm{L}^2(\Omega),$$

so there exists $\hat{\varphi} \in L^2(\Omega)$, such that $\hat{\varphi} \geq 0$ and

$$\hat{\varphi} = \frac{1}{C} A_n \hat{\varphi} + h.$$

Then $\hat{\varphi} \in L\infty_0(\Omega)$, $h, \beta_n \in C_0\infty(\Omega)$ and

$$\|\mathbb{G}_m(\varphi)\|_{L^\infty} \leq m \int_{K_m} \varphi dx \leq m|K_m|^{1/2}\|\phi\|_{L^2}.$$

Let $v_0 = h$, $v_{i+1} = \sup(\frac{1}{C} A_n v_i, h)$, then for any i,

$$v_i \leq v_{i+1} \leq \hat{\varphi},$$

then there exists v_n such that

$$v_i \to v_n \quad \text{as} \quad \mathrm{i} \to +\infty$$

and v_n is the solution of (3.11).

Finally, we prove (3.14) by inductive method, that $r_{n-1} < C$ implies $r_n < C$. The spectral radius $r(\beta) = r(\beta\mathbb{G}_m)$ is continuous and increasing with respect to β. So if $r_n \geq C$, there exists $\beta_* \in L_0^\infty(\Omega)$ such that $\beta_{n-1} \leq \beta_* \leq \beta_n$ and $r(\beta^*) = C$. So there exists $v \in L_0^\infty(\Omega)$ and $v \in \hat{\mathbb{X}}$,

$$\frac{Cv}{\mathbb{G}v} \leq \frac{Cv}{\mathbb{G}_m v} \leq \beta_* \leq j'(u_n).$$

Let

$$u_* = \mu\mathbb{G}_m(u_n\beta_* - j^*(\beta_*)) + \mu f \tag{3.15}$$

and multiply (3.15) by v, we have

$$\begin{aligned} \textstyle\int_\Omega u_* v &= \mu \textstyle\int_\Omega \mathbb{G}_K(u_n\beta_* - j^*(\beta_*))(x)v(y)dxdy + \mu \textstyle\int_\Omega f_n v \\ &\leq C\mu \textstyle\int_\Omega uv - \mu \textstyle\int_\Omega j^*(\beta_*(x)))\mathbb{G}_K(v)(y)dxdy + \mu F_C(v), \end{aligned} \tag{3.16}$$

where $\mathbb{G}_K(h)(x) = \int_\Omega \chi_K(x)\chi_K(y)G(x,y)h(y)dy$. For $F_C(v)$, we have

$$\begin{aligned} F_C(v) &= \textstyle\int_\Omega j^*(\frac{Cv}{\mathbb{G}(v)})\mathbb{G}(v)dx \\ &\leq \textstyle\int_\Omega j^*(\frac{Cv}{\mathbb{G}_K(v)})\mathbb{G}_K(v)dx. \end{aligned} \tag{3.17}$$

Then (3.16) and (3.17) imply that

$$\int_\Omega u_* v \leq \mu C \int_\Omega u_n v,$$

combining $u_* \geq u_n$, we have that

$$\int_\Omega u_* v dx = 0. \tag{3.18}$$

Let $K = \{x \in \Omega, v(x) > 0\}$, then by (3.18), we have

$$\int_K u_* = 0,$$

which, combining (3.15) and $f \geq 0$, implies that

$$0 \geq \mu \int_K dx \int_K \chi_K(x)\chi_K(y) G(x,y)[u_n(y)\beta_*(y) - j^*(\beta_*)(y)]dy.$$

By the fact of $\beta_{n-1} \leq \beta_* \leq \beta_n$,

$$K \subset \{x \in \Omega, \beta_*(x) > 0\} \subset \{x \in \Omega, u_n(y)\beta_*(y) - j^*(\beta_*)(y) > 0\},$$

which implies that

$$G(x,y) = 0 \quad \text{a.e.} \quad \mathrm{K} \times \mathrm{K},$$

then $\mathbb{G}_K(v) = 0$ in K implies that $v \equiv 0$, that is impossible. And we finish the proof. $\square$

For f changing signs, we assume that there exists a measurable function v such that

(*i*) $v \in L^1(K_n)$ and $\mathbb{N}(\cdot,\cdot)j(v) \in L^1(K_n, \Omega)$ for all $n \in \mathbb{N}$;

(*ii*) $v(x) \leq \mathbb{N}(j(v))(x) + f(x)$ a.e. in Ω.

If $j : \Omega \times \mathbb{R} \to (-\infty, \infty]$ measurable function satisfies Conjugate condition. Denote

$$j_v^*(x,r) = \sup_{a \geq v(x)} ra - j(x,a)$$

$$\hat{\mathbb{X}}_v = \{h \in L_c^\infty(\Omega) : \exists C > 1 \text{ s.t. } \mathrm{j}_\mathrm{v}^*(\frac{\mathrm{Ch}}{\mathbb{G}(\mathrm{h})})\mathbb{G}(\mathrm{h}) \in \mathrm{L}^1(\Omega)\}$$

Corollary 3.1 *Assume that f is measurable. Then*

$$\begin{cases} (i)\ u \geq v, \quad u(x) = \mathbb{G}(j(u))(x) + f(x), \quad x \in \Omega, \\ (ii)\ uh \in L^1(\Omega), \quad \forall h \in \hat{\mathbb{X}}_v \end{cases} \tag{3.19}$$

admits a solution, if and only if

$$\int_\Omega fh dx \leq \int_\Omega j_v^*(\frac{h}{\mathbb{G}(h)})\mathbb{G}(h)dx, \quad \forall h \in \hat{\mathbb{X}}_v. \tag{3.20}$$

Proof. Let $w = u - v$, $\tilde{f}(x) = f(x) + \mathbb{G}(j(v))(x) - v(x)$, $\tilde{j}(x,r) = j(x, r + v(x)) - j(x, v(x))$ if $j(x, v(x)) < \infty$ and $\tilde{j}(x,r) = \infty$, if $j(x, v(x)) = \infty$ and $r \geq 0$.

Step 1. (3.19) is equivalent to

$$\begin{cases} (i)\ w \geq 0, \quad w(x) = \mathbb{G}(\tilde{j}(u))(x) + \tilde{f}(x), \quad x \in \Omega, \\ (ii)\ wh \in L^1(\Omega), \quad \forall h \in \hat{\mathbb{X}}, \end{cases} \tag{3.21}$$

where

$$\hat{\mathbb{X}} = \{h \in L_c^\infty(\Omega) : \exists C > 1 \text{ s.t. } \tilde{\mathrm{j}}^*(\frac{\mathrm{Ch}}{\mathbb{G}(\mathrm{h})})\mathbb{G}(\mathrm{h}) \in \mathrm{L}^1(\Omega)\}$$

We have that $\tilde{f} \geq 0$ by condition of v. Now we have to show

$$\hat{\mathbb{X}} = \hat{\mathbb{X}}_v.$$

By directly computation, we have that

$$\tilde{j}^*(x,r) = j^*(x,r) - rv(x) + j(x, v(x)).$$

So

$$\tilde{j}^*(\frac{Ch}{\mathbb{G}(h)})\mathbb{G}(h) = j_v^*\frac{Ch}{\mathbb{G}(h)})\mathbb{G}(h) - Chv + j(x, v(x))\mathbb{G}(h),$$

combining that $hv \in L^1(\Omega)$ and $j(x, v(x))\mathbb{G}(h) \in L^1(\Omega)$, then $\hat{\mathbb{X}} = \hat{\mathbb{X}}_v$.

Step 2. (3.20) is equivalent to

$$\int_\Omega \tilde{f} h dx \leq \int_\Omega \tilde{j}^*(\frac{h}{\mathbb{G}(h)})\mathbb{G}(h) dx, \quad \forall h \in \hat{\mathbb{X}}.$$

In fact, the equivalence derives from

$$\int_\Omega \tilde{f} h dx = \int_\Omega f h dx + \int_\Omega [\mathbb{G}(j(v))(x) - v(x)] h dx$$

and

$$\int_\Omega \tilde{j}^*(\frac{h}{\mathbb{G}(h)})\mathbb{G}(h) dx = \int_\Omega j_v^*(\frac{h}{\mathbb{G}(h)})\mathbb{G}(h) dx + \int_\Omega [\mathbb{G}(j(v))(x) - v(x)] h dx.$$

Now we applying Theorem 3.1 to obtain our corollary. $\square$

4 Proof of Theorem 1.1

In this section, we do the existence of solution to

$$\begin{cases} (-\Delta)^{\alpha} u = u_+^p + \sigma\lambda, & \text{in} \quad \Omega, \\ u(x) = 0, & \text{in} \quad \mathbb{R}^{\mathrm{N}} \setminus \Omega. \end{cases} \tag{4.1}$$

where $p > 1$, $\sigma > 0$ and $\lambda \in \mathfrak{M}(\Omega)$.

Corollary 4.1 *Assume that $p > 1$ $\lambda \in \mathfrak{M}(\Omega)$ and $\sigma > 0$, $\mathbb{G}(\lambda) \in L^1(\Omega)$. Denote $v(x) = \min\{\mathbb{G}(\lambda)(x), 0\}$. Then there exists $u \in L^1_{loc}(\Omega)$ such that $\mathbb{G}(u_+^p) \in L^1_{loc}(\Omega)$ and (4.1) holds in the weak sense of*

$$\begin{cases} u(x) \geq v(x), & x \in \Omega, \\ u(x) = \mathbb{G}(u_+^p)(x) + \mathbb{G}(\lambda)(x), & x \in \Omega, \\ uh \in L^1(\Omega), \qquad h \in \mathbb{X} \end{cases} \tag{4.2}$$

if and only if

$$\sigma \int_\Omega \mathbb{G}(h) d\lambda \leq \frac{p-1}{p^{p'}} \int_\Omega \frac{h^{p'}}{\mathbb{G}(h)^{p'-1}} dx, \quad h \in \mathbb{X}, \tag{4.3}$$

where $p' = \frac{p}{p-1}$ and $\mathbb{X}$ is defined by (3.5).

Proof. We are going to use Corollary 3.1 in this proof. In Corollary 3.1, $\mathbb{X} = \hat{\mathbb{X}}$ and

$$j^*(r) = \begin{cases} \frac{p-1}{p^{p'}} r^{p'}, & \text{if} \quad \mathrm{r} \geq 0, \\ +\infty, & \text{if} \quad \mathrm{r} < 0. \end{cases}$$

By $v(x) = \min\{\mathbb{G}(\lambda)(x), 0\} \leq 0$, then we have $j_v^* = j*$. We note here that $\hat{\mathbb{X}}_v = \mathbb{X}$.

We claim that (3.20) is equivalent to (4.3). In fact, for $h \in \mathbb{X}$,

$$\int_\Omega fh dx = \int_\Omega \int_\Omega G(x, y) h(x) d\lambda(y) dx = \int_\Omega \mathbb{G}(h) d\lambda.$$

Then applied Corollary 3.1 to get our results. $\square$

We note here that Theorem 3.1, Corollary 3.1 and Corollary 4.1 hold for any open smooth domain, including $\Omega = \mathbb{R}^N$. In what follows we do the application of Corollary 4.1 in bounded domain. And the embedding:

$$\mathbb{G} : L^s(\Omega) \to L^r(\Omega)$$

plays an important roles. The precise statement is following:

Lemma 4.1 *Assume that Ω is open, smooth and bounded.*
(i) if

$$\frac{1}{s} < \frac{2\alpha}{N},$$

then there exists some $C > 0$ such that

$$\|\mathbb{G}(h)\|_{L^\infty(\Omega)} \leq C\|h\|_{L^s(\Omega)}; \tag{4.4}$$

(ii) if

$$\frac{1}{s} \leq \frac{1}{r} + \frac{2\alpha}{N} \quad \text{and} \quad \mathrm{s} > 1,$$

then there exists some $C > 0$ such that

$$\|\mathbb{G}(h)\|_{L^r(\Omega)} \leq C\|h\|_{L^s(\Omega)}. \tag{4.5}$$

(iii) if

$$1 < \frac{1}{r} + \frac{2\alpha}{N},$$

then there exists some $C > 0$ such that

$$\|\mathbb{G}(h)\|_{L^r(\Omega)} \leq C\|h\|_{L^1(\Omega)}. \tag{4.6}$$

Proof. **Step 1.** To prove (4.4). By the Hölder inequality and (2.12), for any $x \in \Omega$,

$$\begin{aligned}
\|\int_\Omega G(x,y)h(y)dy\|_{L^\infty(\Omega)} &\leq \|(\int_\Omega G(x,y)^{s'}dy)^{\frac{1}{s'}}(\int_\Omega |h(y)|^s dy)^{\frac{1}{s}}\|_{L^\infty(\Omega)} \\
&\leq C\|h\|_{L^s(\Omega)}\|\int_\Omega \frac{1}{|x-y|^{(N-2\alpha)s'}}dy\|_{L^\infty(\Omega)},
\end{aligned}$$

where $s' = \frac{s}{s-1}$. Since $\frac{1}{s} < \frac{2\alpha}{N}$, that implies $(N-2\alpha)s' < N$, and Ω is bounded, then

$$\begin{aligned}
\int_\Omega \frac{1}{|x-y|^{(N-2\alpha)s'}}dy &\leq \int_{B_D(x)} \frac{1}{|x-y|^{(N-2\alpha)s'}}dy \\
&= C\int_0^D r^{N-1-(N-2\alpha)s'}dr \\
&< CD^{N-(N-2\alpha)s'},
\end{aligned}$$

where $D = \sup\{|x-y| : \ x, y \in \Omega\}$. Then (4.4) holds.

Step 2. To prove (4.5) and (4.6) with $r \le s$. We have

$$
\begin{aligned}
\{\int_\Omega [\int_\Omega G(x,y)h(y)dy]^r dx\}^{\frac{1}{r}} &= \{\int_{\mathbb{R}^N}[\int_{\mathbb{R}^N} G(x,y)h(y)dy]^r dx\}^{\frac{1}{r}} \\
&\le C\{\int_{\mathbb{R}^N}[\int_{\mathbb{R}^N} \frac{h(y)\chi_\Omega(x)\chi_\Omega(y)}{|x-y|^{N-2\alpha}}dy]^r dx\}^{\frac{1}{r}} \\
&\le C\{\int_{\mathbb{R}^N}[\int_{\mathbb{R}^N} \frac{h(x-y)\chi_\Omega(x)\chi_\Omega(x-y)}{|y|^{N-2\alpha}}dy]^r dx\}^{\frac{1}{r}};
\end{aligned}
$$

by using the integral Minkowski's inequality, then,

$$
\begin{aligned}
&\{\int_\Omega [\int_\Omega G(x,y)h(y)dy]^r dx\}^{\frac{1}{r}} \\
&\le C\int_{\mathbb{R}^N}[\int_{\mathbb{R}^N} \frac{h^r(x-y)\chi_\Omega(x)\chi_\Omega(x-y)}{|y|^{(N-2\alpha)r}}dx]^{\frac{1}{r}}dy \\
&\le C\int_{\tilde\Omega}[\int_{\mathbb{R}^N} h^r(x-y)\chi_\Omega(x)\chi_\Omega(x-y)dx]^{\frac{1}{r}}\frac{1}{|y|^{N-2\alpha}}dy \\
&\le C\|h\|_{L^r(\Omega)} \le C\|h\|_{L^s(\Omega)},
\end{aligned}
$$

where $\tilde\Omega = \{x-y, x, y\in\Omega\}$ is bounded.

Step 3. To prove (4.5) and (4.6) with $r > s \ge 1$ and $\frac{1}{s} \le \frac{1}{r} + \frac{2\alpha}{N}$. We claim that if $r > s$ and $\frac{1}{r^*} = \frac{1}{s} - \frac{2\alpha}{N}$, the mapping $h \to \mathbb{G}(h)$ is of weak-type (s, r^*), in the sense that

$$
|\{x\in\Omega : |\mathbb{G}(h)| > t\}| \le (A_{s,r^*}\frac{\|h\|_{L^s(\Omega)}}{t})^{r^*}, \quad h\in L^s(\Omega), \quad \text{all } \ t>0, \quad (4.7)
$$

where constant $A_{s,r^*} > 0$.

Denote for $\nu > 0$,

$$
G_0(x,y) = \begin{cases} G(x,y), & \text{if } \ |\mathrm{x}-\mathrm{y}| \le \nu, \\ 0, & \text{if } \ |\mathrm{x}-\mathrm{y}| > \nu. \end{cases}
$$

and $G_\infty(x,y) = G(x,y) - G_0(x,y)$. Then we have that

$$
|\{x\in\Omega : |\mathbb{G}(h)| > 2t\}| \le |\{x\in\Omega : |\mathbb{G}_0(h)| > t\}| + |\{x\in\Omega : |\mathbb{G}_\infty(h)| > t\}|,
$$

where $\mathbb{G}_0(h)$ and $\mathbb{G}_\infty(h)$ is defined similarly to $\mathbb{G}(h)$.

By Step 2 and the integral Minkowski's inequality, we have

$$
|\{x\in\Omega : |\mathbb{G}_0(h)| > t\}| \le \frac{\|\mathbb{G}_0(h)\|^s_{L^s(\Omega)}}{t^s}
$$

$$\begin{aligned}
&\leq \frac{\| \int_\Omega \chi_{B_\nu(x-y)}\Gamma(x-y)|h(y)|dy\|^s_{L^s(\Omega)}}{t^s}\\
&\leq \frac{[\int_\Omega (\int_\Omega |h(x-y)|^s dx)^{\frac{1}{s}}\Gamma(y)\chi_{B_\nu}(y)dy]^s}{t^s}\\
&\leq \frac{\|h\|^s_{L^s(\Omega)}\|\Gamma\chi_{B_\nu}\|^s_{L^1(\Omega)}}{t^s}
\end{aligned}$$

and

$$\|\Gamma\chi_{B_\nu}\|_{L^1(\Omega)}\int_{B_\nu}|x|^{-N+2\alpha}dx = C_1\nu^{2\alpha}.$$

On the other hand,

$$\begin{aligned}
\|\mathbb{G}_\infty(h)\|_{L^\infty(\Omega)} &\leq \| \int_\Omega \chi_{B_\nu^c}(x-y)\Gamma(x-y)|h(y)|dy\|_{L^\infty(\Omega)}\\
&\leq (\int_\Omega |h(y)|^s dy)^{\frac{1}{s}}\|(\int_\Omega \chi_{B_\nu^c}(x-y)\Gamma(x-y)^{s'}dy)^{\frac{1}{s'}}\|_{L^\infty(\Omega)}\\
&\leq \|h\|_{L^s(\Omega)}\|\Gamma\chi_{B_\nu^c}\|_{L^{s'}(\mathbb{R}^N)},
\end{aligned}$$

where $s' = \frac{s}{s-1}$ if $s > 1$, if not, $s' = \infty$.

Since

$$\|\Gamma\chi_{B_\nu^c}\|_{L^{s'}(\mathbb{R}^N)} = [\int_{\mathbb{R}^N\setminus B_\nu}|x|^{(-N+2\alpha)s'}dx]^{\frac{1}{s'}} = C_2\nu^{2\alpha-\frac{N}{s}},$$

by choosing $\nu = (\frac{t}{C_2\|h\|_{L^s(\Omega)}})^{\frac{1}{2\alpha-\frac{N}{s}}}$, then

$$\|\mathbb{G}_\infty(h)\|_{L^\infty(\Omega)} \leq t,$$

that means

$$|\{x\in\Omega : |\mathbb{G}_\infty(h)| > t\}| = 0.$$

With this ν, we have that

$$|\{x\in\Omega : |\mathbb{G}(h)| > 2t\}| \leq C_1\frac{\|h\|^s_{L^s(\Omega)}\nu^{2s\alpha}}{t^s} \leq C_3(\frac{\|h\|_{L^s(\Omega)}}{t})^{r^*}.$$

The argument of (ii) and (iii) with $r > s$ follows by the Marcinkiewicz Interpolation Theorem. The proof completes. $\square$

Proof of Theorem 1.1. Let $h \in \mathbb{X}$ and w such that

$$h = w^{1/p'}\mathbb{G}(h)^{1/p}, \tag{4.8}$$

where $p' = \frac{p}{p-1}$ if $p > 1$, $p' = \infty$ if $p = 1$.

If

$$\frac{1}{q} + \frac{1}{r} \leq 1 \quad \text{with} \quad \mathrm{r} < \infty \tag{4.9}$$

since $\mathbb{G}(h) \geq 0$, we have

$$\int_\Omega \mathbb{G}(h) d\lambda \leq \int_\Omega \mathbb{G}(h) d\lambda_+ \leq \|\lambda_+\|_{L^q(\Omega)} \|\mathbb{G}(h)\|_{L^r(\Omega)}. \tag{4.10}$$

If $r = \infty$, then

$$\int_\Omega \mathbb{G}(h) d\lambda \leq \lambda_+(\Omega) \|\mathbb{G}(h)\|_{L^\infty(\Omega)}. \tag{4.11}$$

If

$$\begin{array}{ll} \frac{1}{s} \leq \frac{1}{r} + \frac{2\alpha}{N} & \text{with } \ \mathrm{s} > 1 \quad \text{or} \\ 1 < \frac{1}{r} + \frac{2\alpha}{N} & \text{with } \ \mathrm{s} = 1 \quad \text{or} \\ \frac{1}{s} < \frac{2\alpha}{N} & \text{with } \ \mathrm{r} = +\infty, \end{array} \tag{4.12}$$

by (4.8) and Lemma 4.1, for some $C > 0$,

$$\|\mathbb{G}(h)\|_{L^r} \leq C \|h\|_{L^s} \leq C\Big(\int_\Omega w^{s/p'} \mathbb{G}(h)^{s/p} dx\Big)^{1/s}.$$

For $1 < s < \infty$, if

$$s < p', \tag{4.13}$$

one gets

$$\|\mathbb{G}(h)\|_{L^r} \leq C\Big(\int_\Omega w dx\Big)^{s/p'} \Big(\int_\Omega \mathbb{G}(h)^{\frac{sp'}{p(p'-s)}} dx\Big)^{\frac{p'-s}{p's}};$$

and if

$$s = p', \tag{4.14}$$

then

$$\|\mathbb{G}(h)\|_{L^r} \leq C \|\mathbb{G}(h)\|_{L^\infty(\Omega)} \int_\Omega w dx;$$

Then if

$$r \geq \frac{sp'}{p(p'-s)}, \tag{4.15}$$

we derive that

$$\|\mathbb{G}(h)\|_{L^r} \leq C \int_\Omega w dx.$$

Together with (4.10), we have

$$\sigma \int_\Omega \mathbb{G}(h) dx \leq C\sigma \int_\Omega w dx,$$

that is (4.3). We apply Corollary 4.1 to obtain of there exists a weak solution of (1.3).
In case (i), $1 < p < \frac{N}{N-2\alpha}$ implies $p' > \frac{N}{2\alpha}$. Then combining $q = 1$, $r = \infty$ and $s = p'$, (4.9-4.15) hold;
In case (ii), $p > \frac{N}{N-2\alpha}$ and $q \ge \frac{Np}{2\alpha p+N}$. Then take $r = q'$ and $s = p'$, (4.9-4.15) hold;
In case (iii), $p = \frac{N}{N-2\alpha}$ and $q > 1$. Then take $r = \frac{q}{q-1}$ and $s = \frac{qN}{2\alpha(q-1)}$, (4.9-4.15) hold. $\square$

5 The particular case $\lambda = \delta_{x_0}$

In this section, our purpose is to find solutions to (1.3).

We introduce following existence theorem:

Theorem 5.1 *Let $p > 0$ and $\lambda \in \mathfrak{M}(\Omega)$ with $\mathbb{G}(\lambda) \ge 0$.*
Assume that

$$\mathbb{G}\mathbb{G}^p(\lambda) \le C_0\mathbb{G}(\lambda), \quad \text{a.e. in} \ \ \Omega, \tag{5.1}$$

where $C_0 > 0$.
(i) If $p > 1$, for $\sigma \in (0, (\frac{p-1}{p})(\frac{1}{pC_0})^{\frac{1}{p-1}}]$, problem (1.3) admits a positive solution $u \in L^1(\Omega) \cap L^p(\Omega)$ such that

$$\sigma\mathbb{G}(\lambda) + \sigma^p\mathbb{G}\mathbb{G}^p(\lambda) < u(x) < \sigma\mathbb{G}(\lambda) + (\frac{p}{p-1})^p\sigma^p\mathbb{G}\mathbb{G}^p(\lambda). \tag{5.2}$$

(ii) If $p = 1$ and $C_0 < 1$, then problem (1.3) admits a positive solution $u \in L^1(\Omega) \cap L^p(\Omega)$ such that

$$\sigma\mathbb{G}(\lambda) + \sigma\mathbb{G}\mathbb{G}(\lambda) < u(x) < \sigma\mathbb{G}(\lambda) + \frac{\sigma}{1 - C_0}\mathbb{G}\mathbb{G}^p(\lambda). \tag{5.3}$$

(iii) If $p \in (0, 1)$, for any $C_0 < \infty$ and any $\sigma > 0$, problem (1.3) admits a positive solution $u \in L^1(\Omega) \cap L^p(\Omega)$ such that

$$\sigma\mathbb{G}(\lambda) + \sigma^p\mathbb{G}\mathbb{G}(\lambda) < u(x) < \sigma\mathbb{G}(\lambda) + \sigma^p(\sigma^{p-1}C_0 + 1)^{\frac{p}{1-p}}\mathbb{G}\mathbb{G}^p(\lambda). \tag{5.4}$$

Proof. Let

$$u_0 = \sigma\mathbb{G}(\lambda), \quad u_1 = \sigma\mathbb{G}(\lambda) + \sigma^p\mathbb{G}\mathbb{G}^p(\lambda) \tag{5.5}$$

and

$$u_n = \sigma\mathbb{G}(\lambda) + \mathbb{G}(u_{n-1}^p), \quad n \in \mathbb{N}. \tag{5.6}$$

By monotone iteration, see Theorem 4.2 in [19], problem (1.3) admits a solution if there is a super solution $\bar{u}$, that is,

$$\bar{u} \ge \mathbb{G}(\bar{u}^p) + \sigma\mathbb{G}(\lambda) \quad \text{a.e. in } \Omega. \tag{5.7}$$

To this end, let

$$u_t = t\sigma^p \mathbb{G}(\mathbb{G}^p(\lambda)) + \sigma\mathbb{G}(\lambda),$$

by (5.1), then

$$u_t \le (C_0 t\sigma^p + \sigma)\mathbb{G}(\lambda). \tag{5.8}$$

Then by (5.8) and (5.1), there exists $t > 0$ such that

$$\mathbb{G}(u_t^p) + \sigma\mathbb{G}(\lambda) \le (C_0 t\sigma^p + \sigma)^p \mathbb{G}\mathbb{G}^p(\lambda) + \sigma\mathbb{G}(\lambda) \le u_t. \tag{5.9}$$

Then (5.9) holds if there exists $t > 0$ such that

$$(C_0 t\sigma^{p-1} + 1)^p \le t. \tag{5.10}$$

If $p > 1$ and $C_0\sigma^{p-1} \le (\frac{p-1}{p})^{p-1}\frac{1}{p}$, (5.10) holds for $t = (\frac{p}{p-1})^p$.
If $p = 1$ and $C_0 < 1$, (5.10) holds for $t = \frac{1}{1-C_0}$.
If $p < 1$, (5.10) holds for $t = (C_0\sigma^{p-1} + 1)^{\frac{p}{1-p}}$ where $C_0 > 0$. Then we finish the proof. □

Remark 5.1 *The solution v, obtained by the sequence (5.5) and (5.6), is the minimal solution, that is,*

$$u \ge v \quad \text{in} \quad \mathbb{R}^{\mathrm{N}},$$

for any solution u of 1.3.

Remark 5.2 *In the case of $p \in (0,1)$, we observe that in the behavior (5.4),*

$$\sigma^p(\sigma^{p-1}C_0 + 1)^{\frac{p}{1-p}} = (C_0 + \sigma^{1-p})^{\frac{p}{1-p}} > C_0^{\frac{p}{1-p}}$$

and

$$\sigma^p(\sigma^{p-1}C_0 + 1)^{\frac{p}{1-p}} \to C_0^{\frac{p}{1-p}}, \quad \text{as} \quad \sigma \to 0.$$

So the behavior (5.4) is not so sharp.

We note that the domain Ω is not necessary to be bounded in Theorem 5.1. In case of $\alpha = 1$ and $\Omega = \mathbb{R}^N$, it was built the equivalence among (4.3), (5.1) and the Riesz capacity or Bessel capacity, see [1]. The key step is to build the equivalence between (5.1) and

$$\int_{\mathbb{R}^N} |h|^p d\lambda \le C\int_{\mathbb{R}^N} |\Delta h|^p, \quad h \in C_0^\infty(\mathbb{R}^N).$$

However, it is no easy to obtain similarly estimate

$$\int_{\mathbb{R}^N} |h|^p d\lambda \le C\int_{\mathbb{R}^N} |(-\Delta)^\alpha h|^p, \quad h \in C_0^\infty(\mathbb{R}^N)$$

for $\alpha \in (0, 1)$.

In particular, for $\alpha \in (0, 1)$, $\Omega = \mathbb{R}^N$ and $1 < p < N/p$, it was built the equivalence between (5.1) and

$$\lambda(E) \leq C\ cap(E, W^{\alpha,p}),$$

where $C > 0$ and $cap(E, W^{\alpha,p})$ is the Riesz capacity defined by

$$cap(E, W^{\alpha,p}) = \inf\{\|u\|^p_{L^p} : \Gamma * u \geq 1 \text{ on E}, \mathrm{u} \geq 0, \mathrm{u} \in \mathrm{L^p}\}. \tag{5.11}$$

See [16] for details.

Now we consider the application of Theorem 5.1 in bounded domain.

Lemma 5.1 *Suppose that Ω is an open, bounded and smooth domain of $\mathbb{R}^N$, $p > 0$ and $\lambda \in \mathfrak{M}_+(\Omega)$ with $\lambda(\Omega) = 1$. If*

$$p < \frac{N}{N - 2\alpha}, \tag{5.12}$$

then $\mathbb{G}(\lambda) \in L^1(\Omega)$, and there exists $C = C(N, \alpha, \beta, \lambda, \Omega) > 0$ such that

$$\mathbb{G}\mathbb{G}^p(\lambda) \leq C\mathbb{G}(\lambda), \quad \text{a.e. in} \quad \Omega. \tag{5.13}$$

Proof. By Jensen's inequality with $\lambda(\Omega) = 1$,

$$\mathbb{G}^p(\lambda) = [\int_\Omega G(x, y)d\lambda(y)]^p \leq \int_\Omega G^p(x, y)d\lambda(y),$$

which, combining (2.11), implies that

$$\begin{aligned}
\mathbb{G}\mathbb{G}^p(\lambda) &\leq \int_\Omega \int_\Omega G(x, y)G(y, z)G^{p-1}(y, z)d\lambda(z)dy \\
&\leq C \int_\Omega G(x, z) \int_\Omega \frac{1}{|x - y|^{N-2\alpha}} + \frac{1}{|y - z|^{(N-2\alpha)p}} dyd\lambda(z) \\
&\leq C \int_\Omega G(x, z)d\lambda(z) \\
&= C\mathbb{G}(\lambda),
\end{aligned}$$

where $\int_\Omega \frac{1}{|x-y|^{N-2\alpha}} + \frac{1}{|y-z|^{(N-2\alpha)p}} dy$ is bounded by (5.14). $\square$

In particular, if $\lambda = \delta_{x_0}$, the behavior of the solution obtained by Theorem 5.1 is controlled by $\mathbb{G}(\lambda)$ and $\mathbb{G}\mathbb{G}^p(\lambda)$. Therefore, we have to do estimate of the behavior of $\mathbb{G}\mathbb{G}^p(\lambda)$.

Lemma 5.2 *Assume that Ω is an open, bounded and smooth domain of $\mathbb{R}^N$, $x_0 \in \Omega$ and $\lambda = \delta_{x_0}$. If*

$$0 < p < \frac{N}{N-2\alpha}, \tag{5.14}$$

then there exists a positive constant $C = C(N, \alpha, \lambda, \Omega) > 1$ and such that if $p \in (\frac{2\alpha}{N-2\alpha}, \frac{N}{N-2\alpha})$,

$$\frac{1}{C} \le \mathbb{G}\mathbb{G}^p(\lambda)|x - x_0|^{-2\alpha+(N-2\alpha)p} \le C, \quad \text{in} \quad \mathrm{B_r(x_0)} \setminus \{\mathrm{x_0}\}, \tag{5.15}$$

if $p = \frac{2\alpha}{N-2\alpha}$,

$$\frac{1}{C} \le \mathbb{G}\mathbb{G}^p(\lambda)(-\ln|x - x_0|)^{-1} \le C, \quad \text{in} \quad \mathrm{B_r(x_0)} \setminus \{\mathrm{x_0}\}, \tag{5.16}$$

if $p < \frac{2\alpha}{N-2\alpha}$,

$$\frac{1}{C} \le \mathbb{G}\mathbb{G}^p(\lambda) \le C, \quad \text{in} \quad \mathrm{B_r(x_0)} \setminus \{\mathrm{x_0}\}, \tag{5.17}$$

where $r = \frac{\min\{1, d(x_0)\}}{4}$.

Proof. Step 1. the case of $\frac{2\alpha}{N-2\alpha} < p < \frac{N}{N-2\alpha}$. Since $G(x, y) < \Gamma(x - y)$ and $\mathbb{G}(\lambda) = G(x, x_0)$, then for $x \neq x_0$,

$$\begin{aligned}
\mathbb{G}\mathbb{G}^p(x) \ &< C \textstyle\int_{B_D(x_0)} \frac{1}{|y-x|^{N-2\alpha}} \frac{1}{|y-x_0|^{(N-2\alpha)p}} dx \\
&= C \textstyle\int_{B_D(0)} \frac{1}{|x-x_0-y|^{N-2\alpha}} \frac{1}{|y|^{(N-2\alpha)p}} dx \\
&\le C|x - x_0|^{2\alpha-(N-2\alpha)p}(C + \textstyle\int_1^{\frac{D}{|x-x_0|}} s^{2\alpha-1-(N-2\alpha)p} ds) \\
&\le C|x - x_0|^{2\alpha-(N-2\alpha)p},
\end{aligned} \tag{5.18}$$

where $D = \sup\{|x - y|, x, y \in \Omega\} < \infty$.

On the other hand, for $x \in B_r(x_0)$ with $r = d(x_0)/4$,

$$G(x, x_0) > C\Gamma(x - x_0)$$

and

$$\begin{aligned}
\mathbb{G}\mathbb{G}^p(x) \ &\ge C \textstyle\int_{B_r(x_0)} \frac{1}{|y-x|^{N-2\alpha}} \frac{1}{|y-x_0|^{(N-2\alpha)p}} dx \\
&= C \textstyle\int_{B_r(0)} \frac{1}{|x-x_0-y|^{N-2\alpha}} \frac{1}{|y|^{(N-2\alpha)p}} dx \\
&\ge C|x - x_0|^{2\alpha-(N-2\alpha)p}(C + \textstyle\int_1^{\frac{r}{|x-x_0|}} s^{2\alpha-1-(N-2\alpha)p} ds) \\
&\ge C|x - x_0|^{2\alpha-(N-2\alpha)p},
\end{aligned} \tag{5.19}$$

for some $C > 0$.

Step 2. For $p = \frac{2\alpha}{N-2\alpha}$. Then (5.18) becomes

$$\mathbb{G}\mathbb{G}^p(x) \le C + \int_1^{\frac{D}{|x-x_0|}} s^{-1}ds \le C(1 + \ln|x - x_0|)$$

and (5.19) becomes

$$\mathbb{G}\mathbb{G}^p(x) \ge C + \int_1^{\frac{D}{|x-x_0|}} s^{-1}ds \le C(1 + \ln|x - x_0|).$$

Step 3. For $p < \frac{2\alpha}{N-2\alpha}$. We prove that $\int_\Omega \frac{1}{|y-x|^{N-2\alpha}} \frac{1}{|y-x_0|^{(N-2\alpha)p}}dy$ is bounded. Indeed, for $x \in B_r(x_0)$ with $r = \frac{\min\{1,d(x_0)\}}{8}$, by h $\acute{}$older inequality

$$\begin{aligned}
&\int_{B_{2r}(x_0)} \frac{1}{|y-x|^{N-2\alpha}} \frac{1}{|y-x_0|^{(N-2\alpha)p}}dy \\
&\le \int_{B_{2r}(x_0)} [\frac{1}{|y-x|^{(N-2\alpha)(p+1)}} + \frac{1}{|y-x_0|^{(N-2\alpha)(p+1)}}]dy \\
&\le \int_{B_{4r}(x)} \frac{1}{|y-x|^{(N-2\alpha)(p+1)}}dy + \int_{B_{2r}(x_0)} \frac{1}{|y-x_0|^{(N-2\alpha)(p+1)}}dy \\
&< \infty,
\end{aligned}$$

since $(N - 2\alpha)(p + 1) < N$, that is, $p < \frac{2\alpha}{N-2\alpha}$.

□

Proof of Theorem 1.2 and Theorem 1.3. The existence of solution to (1.3) with $\lambda = \delta_{x_0}$ follows Theorem 5.1 and Lemma 5.1 under the assumption of $\sigma > 0$ small enough and $p \neq 1$. Also the behavior of the solution near x_0 should be (5.2) for $p > 1$ and (5.4) for $p \in (0, 1)$. Combining Lemma 5.2 and for $x \in \Omega$, by (2.10)

$$0 < \frac{C(N,\alpha)}{|x - x_0|^{N-2\alpha}} - \mathbb{G}(x, x_0) = \phi(x, x_0) < C(N, \alpha)d(x_0)^{-N+2\alpha},$$

we have that the result. □

We note here that it is not able to assume that $C < 1$ in the estimate (5.13) as the request of Theorem 5.1 for the case of $p = 1$. Therefore, in the following we put some small number Λ as the coefficient of the power source of (1.9), to make that the monotone iteration converges. That is,

Theorem 5.2 *Let $\lambda \in \mathfrak{M}(\Omega)$ with $\mathbb{G}(\lambda) \ge 0$. If there exists some $C_0 > 0$ such that*

$$\mathbb{G}\mathbb{G}(\lambda) \le C_0\mathbb{G}(\lambda), \quad \text{a.e. in } \Omega, \tag{5.20}$$

and $\Lambda C_0 < 1$, *then for any* $\sigma > 0$ *problem (1.9) admits a positive solution* $u \in L^1(\Omega)$ *such that*

$$\sigma\mathbb{G}(\lambda) + \sigma\Lambda\mathbb{G}\mathbb{G}(\lambda) < u(x) < \sigma\mathbb{G}(\lambda) + \frac{\sigma\Lambda}{1-\Lambda C_0}\mathbb{G}\mathbb{G}(\lambda). \tag{5.21}$$

Proof. Let $u_0 = \sigma\mathbb{G}(\lambda)$, $u_1 = \sigma\mathbb{G}(\lambda) + \sigma\mathbb{G}\Lambda\mathbb{G}(\lambda)$ and

$$u_n = \sigma\mathbb{G}(\lambda) + \mathbb{G}(\Lambda u_{n-1}), \quad n \in \mathbb{N}.$$

Proceed as the proof of Theorem 5.1 and (5.10) becomes

$$\Lambda C_0 t + 1 \leq t,$$

which implies the result if $\Lambda C_0 < 1$. $\square$

Proof of Theorem 1.4. The existence of solution to (1.9) with $\lambda = \delta_{x_0}$ follows Theorem 5.2 and Lemma 5.1 under the assumption of $\Lambda > 0$ small enough and $p = 1$. Combining Lemma and Lemma 5.2 and for $x \in \Omega$,

$$0 < \frac{C(N,\alpha)}{|x-x_0|^{N-2\alpha}} - \mathbb{G}(x,x_0) = \phi(x,x_0) < C(N,\alpha)d(x_0)^{-N+2\alpha},$$

we have that the asymptotic behavior.

Prove the uniqueness. Let λ_1 and φ_1 be the first eigenvalue and responding eigenfunction respectively, of

$$\begin{cases} (-\Delta)^\alpha u = \lambda_1 u, & \text{in} \quad \Omega, \\ u(x) = 0, & \text{on} \quad \partial\Omega. \end{cases}$$

Then the coefficient Λ satisfies $\Lambda < \lambda_1$, if not,then by computing directly, we have that

$$\phi_1(x_0) = 0,$$

which is impossible with $\phi_1 > 0$ in Ω.

We know the minimal solution v of (1.9) with $\lambda = \delta_{x_0}$ is obtained by the sequence $u_0 = \sigma\mathbb{G}\delta_{x_0}$ and $u_n = \Lambda\mathbb{G}((u_{n-1})_+) + \sigma\mathbb{G}\delta_{x_0}$ $n \in \mathbb{N}$. Let u be another solution of (1.9) with $\lambda = \delta_{x_0}$, then we have $u \geq v$. We assume that $u \not\equiv v$,

$$0 \leq u - v = \Lambda\mathbb{G}(u-v),$$

which multiples by φ_1 and integrate over Ω to get that

$$\begin{aligned}\int_\Omega (u-v)\varphi_1 &= \Lambda\int_\Omega \varphi_1\mathbb{G}(u-v)\\ &= \Lambda\int_\Omega (u-v)\mathbb{G}(\varphi_1)\\ &= \frac{\Lambda}{\lambda_1}\int_\Omega (u-v)\varphi_1\\ &< \int_\Omega (u-v)\varphi_1,\end{aligned}$$

which is impossible. $\square$

6 Asymptotic behavior of the solutions

In the first of this section, we do some estimate for solutions of (1.3) and some type of uniqueness. Let u and v be two solutions of (1.3) and v be the minimal one obtained in section§5. For $\lambda = \delta_{x_0}$, by regularity result we have know u, v are continuous in $\Omega \setminus \{0\}$. See [10] for the regularity.

Proposition 6.1 *Suppose that Ω is an open, bounded and smooth domain of $\mathbb{R}^N$ $(N \geq 2)$, $0 < p < \frac{N}{N-2\alpha}$ and $\lambda = \delta_{x_0}$ with $x_0 \in \Omega$.*

Let u be a solution of

$$u = \mathbb{G}(u_+^p) + \mathbb{G}(\sigma\delta_{x_0})$$

such that there exists $\tau < \frac{N}{p}$ having

$$\limsup_{x\to x_0} u(x)|x-x_0|^\tau < \infty. \tag{6.1}$$

Then we have that

$$\lim_{x\to x_0} u(x)|x-x_0|^{N-2\alpha} = C(N,\alpha)\sigma. \tag{6.2}$$

Proof. We divide the proof into several steps.
Step 1: there exists (x_n) such that

$$x_n \to x_0 \quad \text{as} \quad \text{n} \to +\infty$$

and

$$\lim_{n\to\infty} \frac{\mathbb{G}(u_+^p)(x_n)}{u(x_n)} = 0. \tag{6.3}$$

By (6.6), there exist $\tau_0 \in [N-2\alpha, \frac{N}{p})$ and $C_1 > 0$ such that

$$u(x)|x-x_0|^{\tau_0} \le C_1, \quad x \in \mathbb{R}^N \tag{6.4}$$

and a sequence (x_n) such that $x_n \to x_0$ as $n \to \infty$ and

$$u(x_n)|x_n-x_0|^{\tau_0-\epsilon} \ge C_2, \tag{6.5}$$

where $C_1, C_1 > 0$ and $\epsilon \in [0, \min\{\tau_0 - N + 2\alpha, \frac{2\alpha+(1-p)\tau_0}{2}\}]$ small enough. Then

$$\begin{aligned} \frac{\mathbb{G}(u_+^p)(x_n)}{u(x_n)} &\le C|x_n-x_0|^{\tau_0-\epsilon}\int_\Omega \frac{1}{|x_n-y|^{N-2\alpha}}\frac{1}{|x_0-y|^{p\tau_0}}dy \\ &\le C|x_n-x_0|^{2\alpha+(1-p)\tau_0-\epsilon} \\ &\to 0 \quad \text{as} \quad \text{n} \to \infty, \end{aligned}$$

since $2\alpha + (1-p)\tau_0 - \epsilon > 0$.
Step 2: to prove

$$\limsup_{x\to x_0} u(x)|x-x_0|^{N-2\alpha} < \infty. \tag{6.6}$$

If (6.4) hold for $\tau_0 > N - 2\alpha$, from Step 1 and

$$1 = \frac{\mathbb{G}(u_+^p)(x_n)}{u(x_n)} + \frac{\mathbb{G}(\sigma\delta_{x_0})(x_n)}{u(x_n)},$$

where we see that

$$\lim_{n\to\infty} \frac{\mathbb{G}(\sigma\delta_{x_0})(x_n)}{u(x_n)} = 1, \tag{6.7}$$

which implies that

$$\lim_{n\to\infty} u(x_n)|x_n-x_0|^{N-2\alpha} = C(N,\alpha)\sigma. \tag{6.8}$$

Now we obtain a contradiction between (6.8) and (6.5). So $\tau_0 = N - 2\alpha$.
Step 3: to prove that

$$\lim_{x\to x_0} u(x)|x-x_0|^{N-2\alpha} = C(N,\alpha)\sigma. \tag{6.9}$$

We see the fact $u(x) \ge \mathbb{G}(\sigma\delta_{x_0})$, which implies that

$$\liminf_{x\to x_0} u(x)|x-x_0|^{N-2\alpha} = C(N,\alpha)\sigma. \tag{6.10}$$

By (6.6) and (6.9),

$$
\begin{aligned}
\frac{\mathbb{G}(u_+^p)(x)}{u(x)} &\le C|x-x_0|^{N-2\alpha}\int_\Omega \frac{1}{|x_n-y|^{N-2\alpha}}\frac{1}{|x_0-y|^{p(N-2\alpha)}}dy \\
&\le C|x-x_0|^{N-p(N-2\alpha)} \\
&\to 0 \quad \text{as} \quad |\mathrm{x}-\mathrm{x}_0|\to 0.
\end{aligned}
\tag{6.11}
$$

Now we assume that there is a sequence (x_n) such that

$$
\lim_{n\to\infty} u(x_n)|x_n-x_0|^{N-2\alpha} > C(N,\alpha)\sigma. \tag{6.12}
$$

From Lemma (6.11) and

$$
1=\frac{\mathbb{G}(u_+^p)(x_n)}{u(x_n)}+\frac{\mathbb{G}(\sigma\delta_{x_0})(x_n)}{u(x_n)},
$$

we see that

$$
\lim_{n\to\infty}\frac{\mathbb{G}(\sigma\delta_{x_0})(x_n)}{u(x_n)}=1, \tag{6.13}
$$

which implies that

$$
\lim_{n\to\infty} u(x_n)|x_n-x_0|^{N-2\alpha} = C(N,\alpha)\sigma. \tag{6.14}
$$

Now we obtain a contradiction between (6.14) and (6.12). □

Finally, we see a weak version of uniqueness for (1.3).

Proposition 6.2 *Suppose that Ω is an open, bounded and smooth domain of $\mathbb{R}^N$ $(N\ge 2)$, $1<p<\frac{N}{N-2\alpha}$ and $\lambda=\delta_{x_0}$ with $x_0\in\Omega$. Assume that v is the minimal solution of (1.3) such that*

$$
v\le c\sigma\mathbb{G}(\delta_{x_0}) \quad \text{in} \quad \mathbb{R}^{\mathrm{N}}
$$

and u is a solution of (1.3) such that

$$
0<u(x)\le C(\sigma)\mathbb{G}(\delta_{x_0})(x), \quad x\in\Omega\setminus\{x_0\}, \tag{6.15}
$$

where $C(\sigma)\to 0$ as $\sigma\to 0$.

If σ is small, then

$$
u\equiv v \quad \text{in} \quad \mathbb{R}^{\mathrm{N}}.
$$

Proof. We assume that
$$u \not\equiv v$$
We know that
$$0 \le u - v = \mathbb{G}[u^p(x) - v^p(x)] \le \mathbb{G}(u^{p-1}(u-v)),$$
and
$$w = u - v \le \mathbb{G}(u^p) \le \frac{C}{|x-x_0|^{(N-2\alpha)p-2\alpha}},$$
where $\frac{(N-2\alpha)p-2\alpha}{N} + \frac{2\alpha}{N} < 1$. Then there exists $r > 1$ such that
$$r[(N-2\alpha)p - 2\alpha] < N \quad \text{and} \quad \frac{1}{\mathrm{r}} + \frac{2\alpha}{\mathrm{N}} < 1.$$

We use Lemma 4.1 with r and $s = \frac{1}{\frac{1}{r}+\frac{2\alpha}{N}} > 1$ to obtain that
$$\|w\|_{L^r(\Omega)} \le \|\mathbb{G}(u^{p-1}w)\|_{L^r(\Omega)} \le c\|u^{p-1}w\|_{L^s(\Omega)}, \tag{6.16}$$
for some constant c independent σ.

By hölder inequality,
$$\|u^{p-1}w\|_{L^s(\Omega)} \le \|u^{p-1}\|_{L^{\frac{sr}{r-s}}(\Omega)}\|w\|_{L^r(\Omega)}, \tag{6.17}$$
where by (6.15),
$$u^{p-1} \le \frac{cC(\sigma)^{p-1}}{|x-x_0|^{(N-2\alpha)(p-1)}},$$
and it follows from $\frac{sr}{r-s} = \frac{N}{2\alpha}$ and $(N-2\alpha)(p-1) < 2\alpha$ that
$$(N-2\alpha)(p-1)\frac{N}{2\alpha} < N,$$
then we have that
$$\|u^{p-1}\|_{L^{\frac{sr}{r-s}}(\Omega)} \le cC(\sigma)^{p-1}\int_\Omega \frac{1}{|x-x_0|^{(N-2\alpha)(p-1)\frac{N}{2\alpha}}}dx \le cC(\sigma)^{p-1}, \tag{6.18}$$
where $c > 0$ independent of σ.

From (6.17), (6.18) and (6.16), we have that
$$\|w\|_{L^r(\Omega)} \le cC(\sigma)^{p-1}\|w\|_{L^r(\Omega)},$$
which is impossible if $cC(\sigma)^{p-1} < 1$ and $\|w\|_{L^r(\Omega)} \ne 0$. $\square$

For $p \in (0,1)$, we have following results:

Theorem 6.1 *Under the hypothesis of Theorem 5.1, we assume that* $p \in (0,1)$ *and* w_0 *is the positive solution of*

$$\begin{cases} (-\Delta)^\alpha u = u^p, & \text{in} \quad \Omega, \\ u(x) = 0, & \text{in} \quad \mathbb{R}^{\mathrm{N}} \setminus \Omega. \end{cases} \tag{6.19}$$

Then for any $\sigma > 0$, *problem (1.3) admits a solution* $u \in L^1(\Omega) \cap L^p(\Omega)$ *such that*

$$w_0 + \sigma\mathbb{G}(\lambda) \le u \le w_0 + \sigma\mathbb{G}(\lambda) + \sigma^p(\sigma^{p-1}C_0 + 1)^{\frac{p}{1-p}}\mathbb{G}\mathbb{G}^p(\lambda) \quad \text{in} \ \ \mathbb{R}^{\mathrm{N}}, \tag{6.20}$$

where $C \ge C_1$ *and* v *is the minimal solution of (1.3).*

Proof of Theorem 6.1. Let

$$u_0 = w_0 \quad \text{and} \quad \mathrm{u}_1 = \sigma\mathbb{G}(\lambda) + \mathbb{G}(\mathrm{u}_0^{\mathrm{p}}) \tag{6.21}$$

and

$$u_n = \sigma\mathbb{G}(\lambda) + \mathbb{G}(u_{n-1}^p), \quad n \in \mathbb{N}. \tag{6.22}$$

We first to prove that

$$u_{n+1} \ge u_n, \quad n \in \mathbb{N}, \ x \in \mathbb{R}^N.$$

We observe that

$$\begin{aligned} u_1 &= \mathbb{G}(w_0^p) + \sigma\mathbb{G}(\lambda) \\ &= w_0 + \sigma\mathbb{G}(\lambda) \\ &\ge u_0. \end{aligned}$$

We assume $u_n \ge u_{n-1}$, then we prove that $u_{n+1} \ge u_n$ by the fact of

$$u_{n+1} - u_n = \mathbb{G}(u_n^p - u_{n-1}^p) \ge 0.$$

problem (1.3) admits a solution generated by u_n defined (6.21) and (6.22) if there is a super solution $\bar{u}$, that is,

$$\bar{u} \ge \mathbb{G}(\bar{u}^p) + \sigma\mathbb{G}(\lambda) \quad \text{a.e. in } \Omega. \tag{6.23}$$

To this end, let

$$u_t = w_0 + t\sigma^p\mathbb{G}(\mathbb{G}^p(\lambda)) + \sigma\mathbb{G}(\lambda),$$

by (5.1), then

$$u_t \le w_0 + (C_0 t\sigma^p + \sigma)\mathbb{G}(\lambda). \tag{6.24}$$

Then by (6.24) and (5.1), there exists $t > 0$ such that

$$\mathbb{G}(u_t^p) + \sigma\mathbb{G}(\lambda) \le (C_0 t\sigma^p + \sigma)^p \mathbb{G}\mathbb{G}^p(\lambda) + \sigma\mathbb{G}(\lambda) \le u_t, \tag{6.25}$$

where we use the fact of

$$(a+b)^p \le a^p + b^p, \quad a, b > 0, \; p \in (0,1).$$

Then (6.25) holds if there exists $t > 0$ such that

$$(C_0 t\sigma^{p-1} + 1)^p \le t. \tag{6.26}$$

Since $p < 1$, (6.26) holds for $t = (C_0\sigma^{p-1} + 1)^{\frac{p}{1-p}}$ where $C_0 > 0$. Then we finish the proof. $\square$

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